Exact localized solution for nonconservative systems with delayed nonlinear response

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We found an exact solitonlike solution for systems with gain and loss and delayed nonlinear response. An example of an application of this solution is the passively mode-locked laser with slow saturable absorber. [S1063-651X(98)11502-7]

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Localized structures are objects that define, to a large extent, the general dynamics of dissipative systems far from equilibrium. These are known as ''solitary waves'' or ''solitons'' in a broad sense. Properties of these waves in various conservative and nonconservative systems attracted a great deal of attention in recent years. The passively mode-locked laser [1–7] is an example of a dissipative system far from equilibrium. Ultrashort pulse generation in laser systems is based on a variety of schemes including figure-eight fiber laser design [1], fast saturable absorber [5], coupled cavity [6], additive pulse mode locking [7], nonlinear polarization rotation [8,9], and stretched pulse operation [10]. The use of a semiconductor mirror-saturable absorber with relatively slow response time and even slower recovery time has been suggested for a passively mode-locked soliton laser [11]. Experimental verification of this possibility has been reported in [12,13]. Some design aspects of the semiconductor saturable absorber mirrors are given, e.g., in [14]. In the original works [12,13] the pulses were approximated by the solitons of the nonlinear Schrödinger equation. However, in systems with gain and loss this approximation is too rough and is close to reality for a very limited range of parameters. In the present work we have found the exact solitonlike solution for pulses generated by a solid-state laser with a semiconductor saturable absorber with slow recovery time when the pulse amplitude is much smaller than the saturation threshold of the absorber.

Despite the fact that lumped effects are present in the laser, it can be modeled as a distributed system if the changes to the field per round trip are small. The pulse evolution is then governed by the modified NLSE with nonlinear nonconservative terms [4,11,15]:

$$i\psi_z+\frac{D}{2}\psi_{tt}+|\psi|^2\psi=i[g(Q)-\delta_s(|\psi|^2)]\psi+i\beta\psi_{tt}, \quad (1)$$

where z is the cavity round-trip number, t is the retarded time, ψ is the normalized envelope of the optical field, D is the group velocity dispersion coefficient, β stands for spectral filtering ($\beta>0$), $g(Q)$ is the gain in the cavity which depends on the total energy, $Q=\int_{-\infty}^{\infty}|\psi|^2 dt$, of the pulse in one round trip, and $\delta_s(|\psi|^2)$ is the total loss including loss in the semiconductor saturable absorber.

The gain term $g(Q)$ in Eq. (1) describes a gain medium with a recovery time much slower than the round-trip time of the cavity and does not depend explicitly on t. It describes depletion of the gain medium and depends on the total pulse energy

$$g(Q)=\frac{g_0}{1+Q/E_L}, \quad (2)$$

where g_0 is the small signal gain and E_L is the saturation energy. The value of $g(Q)$ decreases with the pulse energy so that within each round trip the pulse energy is limited.

The absorption in the semiconductor can be described by the rate equation

$$\frac{\partial\delta_s}{\partial t}=-\frac{\delta_s-\delta_0}{T_1}-\frac{|\psi|^2}{E_A}\delta_s, \quad (3)$$

where T_1 is the recovery time of the saturable absorber, δ_0 is the loss introduced by the absorber in the absence of pulses, and E_A is the saturation energy of the absorber.

The solution of Eq. (3) can be written in general form:

$$\delta_s(t)=\delta_0\left[\frac{1}{T_1}\int\left\{\exp\left[\int\left(\frac{1}{T_1}+\frac{|\psi|^2}{E_A}\right)dt\right]\right\}dt+1\right]$$
$$\times\exp\left[-\int\left(\frac{1}{T_1}+\frac{|\psi|^2}{E_A}\right)dt\right]. \quad (4)$$

However, this expression does not allow the general solution of Eq. (1) to be found.

We now consider the limiting case when the pulse amplitude is well below the saturation level. The gain coefficient g is constant if we deal with stationary solutions of Eq. (1) when Q is constant. We also assume that the relaxation time is large in comparison to the pulse width. In this case, $T_1\to\infty$, and the loss changes across the pulse are given by the approximate formula

$$\delta_s(t)=\delta_0 \exp\left[-\int \left(\frac{|\psi|^2}{E_A}\right)dt\right]\approx \delta_0-\alpha\int_{-\infty}^{t}|\psi|^2dt+\cdots, \tag{5}$$

where $\alpha=\delta_0/E_A$. Substituting this into Eq. (1) we obtain the equation

$$i\psi_z+\frac{D}{2}\psi_{tt}+|\psi|^2\psi=i\delta\psi+i\beta\psi_{tt}+i\alpha\psi\int_{-\infty}^{t}|\psi|^2dt, \tag{6}$$

where $\delta=g-\delta_0$. This equation is similar to the complex Ginzburg-Landau equation (CGLE) [16,17], except for the nonconservative nonlinear term in the right-hand-side of Eq. (6), which is nonlocal in time. This equation has been studied in a number of publications related to various physical situations [18–20]. Exact solution of Eq. (6) without a spectral filtering term has been presented in [19] and investigated numerically in [20]. It has also been shown [18] that Eq. (6) has a pulselike solution—''autosoliton'' but exact solution has not been found. It is clear though that an explicit form of the solution is very important for its further analysis.

We have found that Eq. (6) has the exact solution, which is the soliton, moving with velocity V:

$$\psi=[A(t-Vz)]^{1+id}e^{iKt-i\omega z}, \tag{7}$$

where

$$A(x)=\frac{A_0}{\cosh(\gamma x)}$$

$$d=-\frac{3D}{4\beta}+\sqrt{\frac{9D^2}{16\beta^2}+2},$$

$$\gamma=\frac{3\alpha d(4\beta^2+D^2)}{4\beta(2\beta-Dd)}\pm\sqrt{\left[\frac{3\alpha d(4\beta^2+D^2)}{4\beta(2\beta-Dd)}\right]^2+\frac{2(\delta-\beta K^2)}{2\beta-dD}}, \tag{8}$$

$$A_0=\gamma\sqrt{3d\left(\beta+\frac{D^2}{4\beta}\right)},\quad K=-\frac{3\alpha d^2(4\beta^2+D^2)}{8\beta^2(1+d^2)},$$

$$V=K\left(D-\frac{2\beta}{d}\right),\quad \omega=\frac{D}{2}(K^2-\gamma^2+d^2\gamma^2)-2\beta d\gamma^2.$$

For obtaining the solution, we used a method similar to the one presented in [16] (see Chap. 13). All the parameters of this solution including the velocity V are fixed and depend on the parameters of the equation. However, there can be two branches of the solution given by the two signs in Eq. (8). An example of the solution for certain values of parameters δ, β and α is shown in Fig. 1. As can be seen from Fig. 1, the soliton always sticks to the gradient of the absorption curve $\delta(t)$. Different values of loss or gain at different sides of the soliton cause it to move relative to the reference frame. The solution disappears for $\delta\to 0$ where its amplitude goes to zero. Note that δ is equal to the amount of loss (or gain) at the left hand side of the pulse. The properties of the solution vary depending on whether the parameter δ is positive or negative.

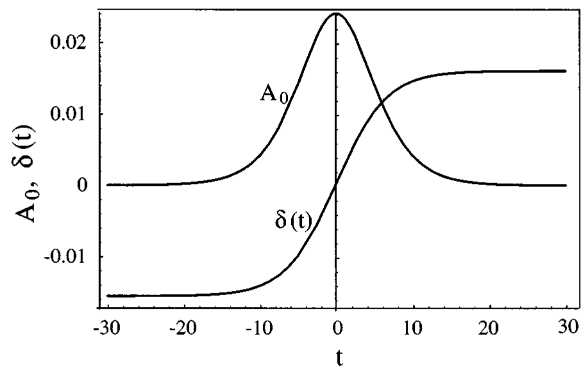

FIG. 1. Soliton profile (solid lines) and the loss curve $\delta(t)=\delta+\alpha\int_{-\infty}^{t}|\psi|^2dt$ (dotted line) defined by the exact solution Eq. (7) for $\delta=-0.015$, $\alpha=0.1$, $\beta=0.02$, and $D=+1$.

Let first δ be negative. In this case, D must be positive and we have to choose the minus sign in front of the square root in the expression for γ. The sign in front of the square root in the expression for d is always positive. The soliton exists in a certain range of parameters. The limits of existence are defined by the nonequality

$$\left[\frac{3\alpha d(4\beta^2+D^2)}{4\beta(2\beta-Dd)}\right]^2+\frac{2(\delta-\beta K^2)}{2\beta-dD}>0. \tag{9}$$

Figure 2 shows the region where this inequality is valid and hence, the solution (7) exists (shaded area). Parameters of the solution versus parameters of the equation are shown in Figs. 3 and 4.

Dependence of the soliton amplitude A_0 on the four parameters of the equation is shown in Fig. 3. The dashed line shows the limits in the parameter space defined by Eq. (9) (boundary of the shaded area). This means that the soliton exists only at positive $0<\beta<\beta_{cr}$ and negative $\delta>\delta_{cr}$ where β_{cr} and δ_{cr} correspond to the edges of shaded area in Fig. 2. Parameters α and D are bounded from below. The amplitude is finite in the above range of parameters and has an upper limit. An important parameter for chirped pulses is the amplitude-width product p/γ. It does not depend on α or δ but weakly depends on β and D as shown in Fig. 4. The velocity of the soliton V does not depend directly on δ but depends linearly on α because K depends linearly on α. The velocity increases with increasing β and D. The velocity is always positive such that the soliton moves in the direction of higher gain.

Now, let us consider the case when δ is positive. In this case the solution exists for both signs in the expression for γ. Hence, we have simultaneously two solutions for the same set of parameters. Below, we restrict ourselves to the case of positive sign in Eq. (8). Moreover, the solution exists for both normal and anomalous dispersion (negative and positive D). This is not surprising [21], because in systems with gain and loss, the pulse is the result of balance not only of the dispersion and nonlinearity (which is impossible at negative D) but also the result of balance between gain and loss. However, for negative D the width of the pulse is much greater than for the positive D (anomalous dispersion) case at the same values of other parameters. Dependence of the soliton amplitude A_0 on the four parameters of the equation

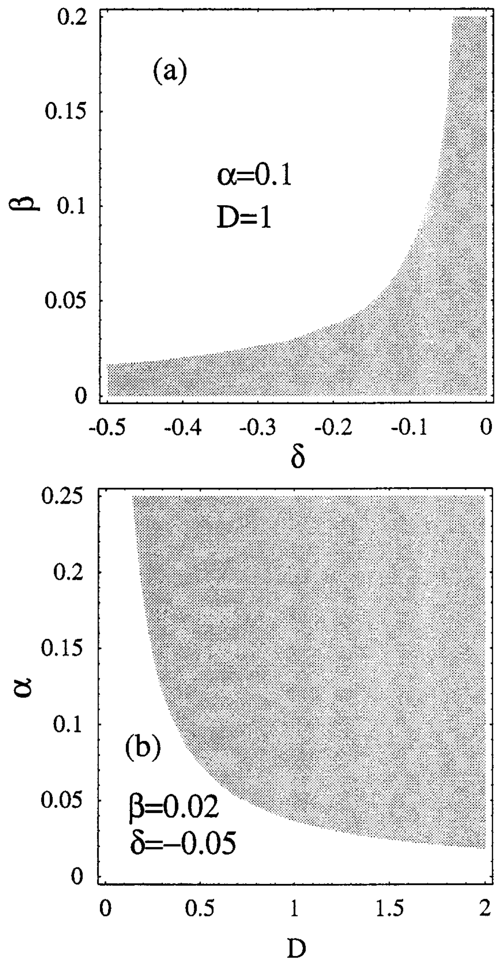

FIG. 2. The space of parameters (a) δ and β and (b) D and α where soliton solution (7) exists. Shaded area is defined by the inequality (9). Parameters of calculation are shown in the plot.

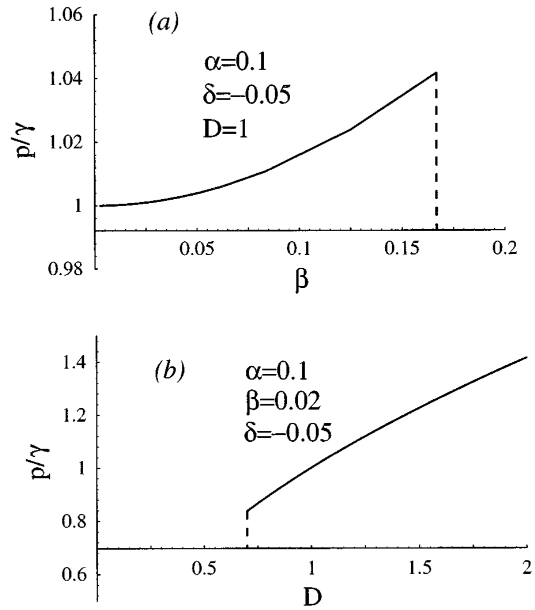

FIG. 4. Amplitude-width product p/γ of the soliton vs (a) β and (b) D. Paramaters of the calculation are given in the plot.

in the case of positive δ is shown in Fig. 5. The amplitude increases to infinity when β decreases to zero. This shows that spectral filtering is crucial for the existence of the pulse in this case. We can also see that the solution exists for both signs of α except for a certain point where the amplitude is zero. The amplitude also goes to zero when $\delta \to 0$. The velocity of the pulse is again positive for both signs of D.

For applications one of the most important properties of the pulses is their stability. Clearly the background is unstable because of the positive gain on one (negative δ) or both (positive δ) sides of the pulse. Total gain is always positive on the right hand side of the pulse. This region is unstable with respect to both generation of continuum and new pulses. However, for negative δ, if we take into account a finite relaxation time this unstable region is finite and generation of new pulses can be controlled by the depletion of the gain medium.

Most importantly the pulse itself is always unstable when the parameters in the equation (6) including δ are fixed. In fact, any increase of the amplitude of the pulse relative to the exact solution increases the total gain across the pulse and

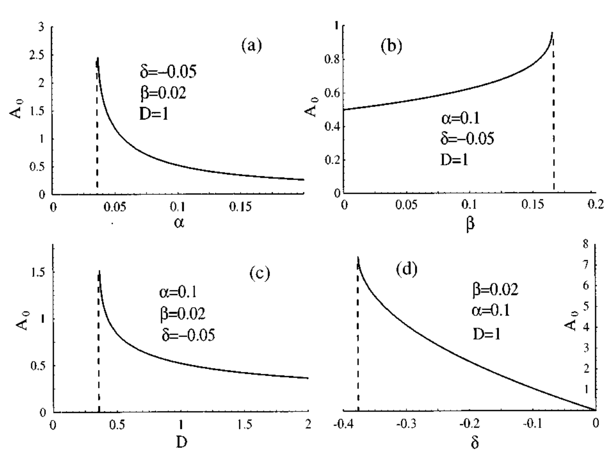

FIG. 3. Dependence of the pulse amplitude on the parameters of the equation (a) α, (b) β, (c) D, and (d) δ (case of negative δ).

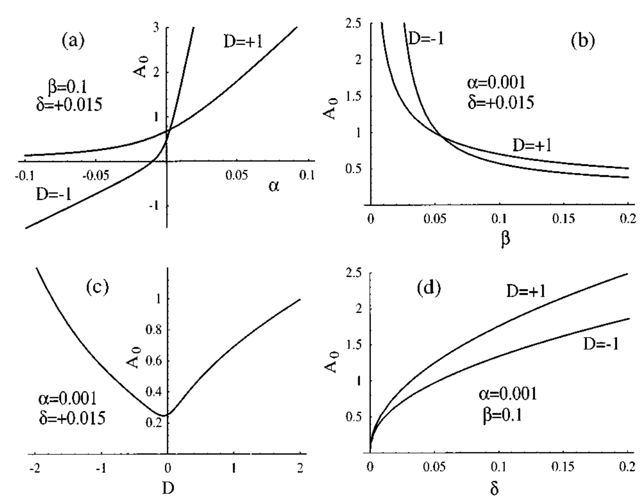

FIG. 5. Dependence of the pulse amplitude on the parameters of the equation α, (b) β, (c) D, and (d) δ (case of positive δ).

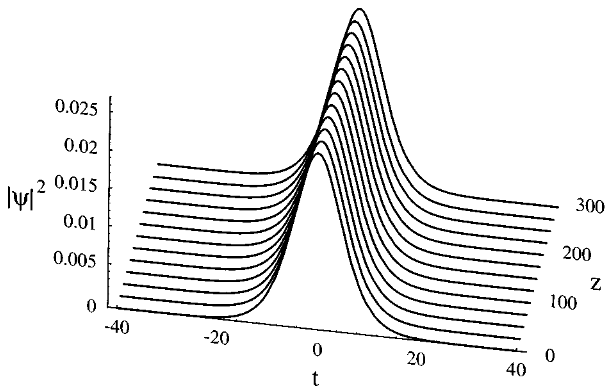

FIG. 6. Stable propagation of the soliton solution for $\delta = -0.015$, $\alpha = 0.1$, $\beta = 0.02$, $D = +1$, $g_0 = 5$, $E_L = 0.00527$, $\delta_0 = 0.1$.

the amplitude increases exponentially. Any decrease of the pulse amplitude works in the opposite direction and the pulse decays. However, for a proper choice of the parameters, the pulse may become stable if δ depends on the total energy of the pulse Q as in Eq. (2). The δ dependence of Q serves as a feedback mechanism that stabilizes the pulse for a certain range of values of g_0, E_L, and δ_0. The feedback mechanism apparently has a delay of at least one round trip time and is defined by the relaxation time of the gain medium. Stability may also depend on this delay time.

Using the exact solution, the pulse energy can easily be calculated as $Q = 2A_0^2/\gamma = 6d\gamma(\beta + D^2/4\beta)$. Substituting $Q = Q(\delta)$ into Eq. (2) we have the equation

$$\frac{g_0}{1 + Q/E_L} - \delta_0 = \delta, \tag{10}$$

which gives stationary values of the parameter δ. The existence and the number of solutions of Eq. (10) depend on the values of the parameters g_0, E_L, and δ_0. Stability of these stationary solutions depends on many parameters and is still an open question. Each particular case can be checked numerically. A numerical simulation showing stable propagation of the pulse for a certain choice of parameters is shown in Fig. 6.

Taking into account the relaxation mechanism in Eq. (4) does not modify the pulse drastically unless the relaxation time T_1 is comparable to the width of the soliton γ^{-1}. On the other hand, for negative δ, the role of the relaxation is to stabilize the background by returning the system to net loss after the pulse has passed.

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